Analysis of anisotropic sector with a radial crack under anti-plane shear loading

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In this paper, the anti-plane shear deformation of an anisotropic sector with a radial crack is investigated. The traction–traction boundary conditions are imposed on the radial edges and the traction-free condition is considered on the circular segment of the sector. A novel mathematical technique is employed for the solution of the problem. This technique consists of the use of some recently proposed finite complex transforms (Shahani, 1999), which have complex analogies to the standard finite Mellin transforms of the first and second kinds. However, it is essential to state the traction-free condition of the crack faces in the form of a singular integral equation which is done in this paper by describing an exact analytical method. The resultant dual integral equations are solved numerically to determine the stress intensity factors at the crack tips. In the special cases, the obtained results coincide with those cited in the literature.

1. Introduction

The stress analysis in a sector with infinite radius has been considered by various investigators. Some authors devoted efforts on deriving the full-field stress distribution in the sector and others investigated the stress singularities near the apex of the sector. Trantir (1948), by employing the Airy stress function and using the Mellin transform, solved the plane elasticity problem of an infinite isotropic wedge. Then, Williams (1952) studied the stress singularities at the wedge apex using the eigenfunction expansion method. Later on, in a series of papers, Bogy (1972) and Kuo and Bogy (1974a,b) employed a complex function representation of the solution in conjunction with a generalized Mellin transform to analyze stress singularities in an anisotropic wedge. Ma and Hour (1989) studied the asymptotic behaviour of the stress components in the vicinity of the apex of a bi-material wedge. They restricted themselves to the derivation of the equation of the poles in the Mellin transform domain and analytical relations for the orders of stress singularities in special cases. The stress distribution in a wedge with finite radius subjected to anti-plane shear deformation was obtained by Kargarnovin et al. (1997). They also extracted the order of stress singularities at the wedge apex as a function of apex angle under different boundary conditions. Afterwards, Shahani (1999), by defining some complex integral transformations, solved the anti-plane deformation problem of anisotropic finite wedges. The traction-free condition was imposed on the circular segment of the wedge. Three different cases of boundary conditions on the radial edges were considered, which are: traction–displacement, displacement–displacement and traction–traction. Several complex transformations were defined to formulate the problem in each of the three cases of the problem corresponding to the radial boundary conditions, separately. These transformations were then related to integral transforms which were complex analogies to the standard finite Mellin transforms of the first and second kinds.

The problem of cracked isotropic wedges under anti-plane shear deformation has been under consideration in the literature. An analytical approach to the problem, under anti-plane shear loading, was presented by Erdogan and Gupta (1975). Shahani and Adibnazari (2000) studied the problem of anti-plane shear deformation of perfectly bonded wedges as well as bonded wedges having infinite radii with an interface crack by means of the Mellin transform. Shahani (2003) derived analytical expressions for the mode III stress intensity factor (SIF) of circular shafts with edge cracks, bonded half planes containing an interfacial edge crack, bonded wedges with an interfacial edge crack and also DCB’s with different boundary conditions. Anti-plane stress analysis for an infinite isotropic wedge weakened with a screw dislocation was accomplished by Faal et al. (2004) using the Mellin transform. The problem of finite annular dissimilar composite wedges with equal apex angles subjected to anti-plane concentrated loadings was considered by Lin and Ma (2004).

The anti-plane shear deformation of a bi-material wedge with finite radius was studied by Shahani (2005) for various boundary conditions. The solution of governing differential equations was accomplished by means of finite Mellin transforms. In part II of the paper, explicit expressions were derived for anti-plane shear
deformation of bi-material circular media containing an interfacial edge crack. Shahani (2006) considered the anti-plane shear deformation problem of two edge-bonded dissimilar isotropic wedges when the sum of the two apex angles is equal to 2π. In this case the problem reduces to that of two edge-bonded dissimilar materials with an interfacial crack subjected to concentrated anti-plane shear tractions on the crack faces. Faal et al. (2007) presented the solution of a Volterra-type screw dislocation in an isotropic finite wedge weakened by multiple cavities under various boundary conditions using the image method. Shahani (2007) extracted closed form solutions for the stress distribution in the isotropic finite wedge under anti-plane deformation. These closed forms have the advantages of showing the possible geometric stress singularity as well as the load singularity explicitly, in addition to the continuity or discontinuity as well as the convergence of the results in the entire region. And also, the stress intensity factors were extracted in the special case of a circular shaft containing an edge crack under different boundary conditions. Chen et al. (2009) analyzed the problem of composite finite wedge under anti-plane shear applied on the circular arc. The considered conditions on the radial edges were free-free, free-fixed, and fixed-fixed. Shahani and Ghadiri (2009) studied the anti-plane shear deformation of bonded isotropic finite wedges with an interface crack. The traction-free condition was imposed on the circular segment of the wedge. Boundary conditions on the radial edges were considered as traction–traction.

Unlike the analytical and practical interest, the problem of bonded anisotropic sectors with an interface crack has not been treated, because of analytical difficulties. The analysis of anisotropic sector with a radial crack under anti-plane shear loading is the subject of the present investigations. The anti-plane shear tractions act on the edges of the bi-material sector and a traction-free condition is prescribed on the arc segment of the sector. The tractions are assumed to act concentrically which allows the solutions to be used as the Green’s function for the analysis of a sector under general distribution of traction. The major task of this paper is to express the prescribed boundary condition on the crack region in the form of a singular integral equation. For this purpose, a novel mathematical technique is employed for the solution of the problem. This technique consists of the use of some recently proposed finite complex transforms (Shahani, 1999). To formulate the mixed boundary value problem corresponding to the presence of the crack, the attempt is made to extract an equation in the form of a standard singular integral equation which is done in this paper by describing an exact analytical method. The resultant singular integral equation is solved numerically to determine the stress intensity factors at the crack tips. In the special cases, the obtained results coincide with those published in the literature.

2. Problem formulation and solution

A bi-material sector with radius a composed of two bonded anisotropic sectors with apex angles θ1 and θ2 and infinite length in the direction perpendicular to the plane of the sector is considered as shown in Fig. 1. Because of imperfect bonding, a crack exists along the common edge. Choosing the common edge as the reference axis for defining the coordinate θ, the crack lies on the line θ = 0 between the radii r = c1 and r = c2. The condition of anti-plane shear deformation is imposed on the sector. The traction–traction boundary conditions are assumed to act on the edges of the sector, however, on the faces of the crack traction-free condition is applied. In such conditions, the only non-zero displacement component is the out-of-plane component, W, which is a function of in-plane coordinates r and θ. Therefore, the nonvanishing stress components are τrz(r, θ) and τrz(r, θ). The constitutive equations for anisotropic materials undergoing anti-plane deformation reduce to

![Fig. 1. Schematic view of a bi-material sector with an interface crack.](image-url)
where \( C_{44}, C_{45} \) and \( C_{55} \) are the elastic constants of the sector and the superscript \( k \) denotes each of the two sectors \( k = 1, 2 \).

In the absence of body forces, by making use of (1), the equilibrium equation in terms of displacement appears as

\[
C_{55} \frac{\partial^2 W^k}{\partial x^2} + 2C_{45} \frac{\partial^2 W^k}{\partial x \partial y} + C_{44} \frac{\partial^2 W^k}{\partial y^2} = 0.
\]

The differential Eq. (2) must be solved under the following boundary conditions:

\[
\begin{align*}
\tau_{12}^k(r, \theta_1) &= P \delta(r - h), \\
\tau_{22}^k(r, -\theta_1) &= P \delta(r - h), \\
W^k(r, 0) &= W^k(r, 0), & 0 \leq r \leq c_1, & c_2 \leq r \leq a, \\
\tau_{12}^k(r, 0) &= \tau_{22}^k(r, 0), \\
\tau_{12}^k(r, 0) &= \delta(r - h), & c_1 \leq r \leq c_2.
\end{align*}
\]

In relations (3) and (4), \( \delta \) denotes the Dirac-Delta function. It is worth mentioning that the choice of these two boundary conditions, leads to the Green’s function solution for the problem. Also, in Eqs. (3) and (4), \( h \) is the location of the application of the concentrated tractions which may vary from zero to \( r = a \). Without loss of generality of the problem, it is supposed that \( c_1 \leq c_2 \leq h \).

The boundary data prescribed on the circular segment of the sector circumference is the traction-free condition, i.e.,

\[
\tau_{12}^k(a, \theta) = 0,
\]

where \( \tau_{12} \) and \( \tau_{22} \) are the stress components in the polar coordinates \( (r, \theta) \).

The solution of the differential Eq. (2) may be performed in the complex plane

\[
Z = x + py
\]

such that

\[
W^k(x, y) = 2 \text{Re}[U^k(Z)] = U^k(Z) + \overline{U^k}(\overline{Z}),
\]

where \( U \) is an arbitrary function of \( Z, \overline{U}(\overline{Z}) \) is the complex conjugate of \( U(Z) \), and \( p \) is a parameter whose value depends on the elasticity constants. Now, it is desired to solve this challenging mixed boundary value problem. To this end, a procedure is adopted based on the constants. Now, it is desired to solve this challenging mixed boundary value problem. To this end, a procedure is adopted based on the rest boundary conditions. Making it possible to apply the boundary conditions of (3) and (4) and then substituting the resulting values is an arbitrary function of \( \theta_1 \).

Substituting (19) and (20) into (21) and facilitating terms, yield

\[
A^2 = \frac{H(-\theta_2)H(\theta_1) - H(-\theta_2)}{H(-\theta_2) - H(-\theta_2)} \frac{a_{25}}{h^2} P \frac{1}{iC_0S}.
\]

Substituting (22) into (19) and facilitating terms, yield

\[
\overline{A}^2 = \frac{H(-\theta_2)H(\theta_1) - H(-\theta_2)}{H(-\theta_2) - H(-\theta_2)} \frac{a_{25}}{h^2} P \frac{1}{iC_0S}.
\]

In relations (20), (22) and (23), the term \( \overline{A}^2(S) \) is the only unknown function, which is determined by applying the rest boundary conditions. Making it possible to apply the boundary conditions (5) and (7), the following unknown function may be defined (Erdogan and Gupta, 1975; Shahani and Adibnazar, 2000):

\[
f(r) = \frac{\partial}{\partial r} \left[ W^k(r, 0) - W^k(r, 0) \right].
\]

With this definition, the condition of continuity of displacements outside the crack, Eq. (5), becomes:

\[
f(r) = 0, \quad 0 \leq r \leq c_1, \quad c_2 \leq r \leq a.
\]
Also, the single-valuedness condition of displacements requires that
\[ \int_{c_1}^{c_2} f(r) \, dr = 0. \quad (26) \]

Rewriting (15) for the two sectors, one can obtain:
\[ W_1^{1}(S, \theta) = \frac{A_1^{1}(S)}{H(0)} + \frac{T_1^{1}(S)}{H(0)} - \frac{2a^2}{S} W_1(a, \theta), \quad (27) \]
\[ W_2^{1}(S, \theta) = \frac{A_2^{1}(S)}{H(0)} + \frac{T_2^{1}(S)}{H(0)} - \frac{2a^2}{S} W_2(a, \theta). \quad (28) \]

Now, taking the inverse transform from relations (27) and (28) and then substituting the resulting expressions into (24), yield
\[ \mathcal{I}(r) = \frac{1}{2\pi i} \int_{c_1}^{c_2} S^2 W_1^{1}(S, 0) \, dS - \frac{1}{2\pi i} \int_{c_1}^{c_2} S^2 W_2^{1}(S, 0) \, dS. \quad (29) \]

Taking the Mellin transform of first kind from (29), results in
\[ \int_{0}^{\infty} \left( \frac{a^2}{S^2} - y^{3-1} \right) v(f) \, dv = -2W_1^{1}(S, 0) + S W_2^{1}(S, 0). \quad (30) \]

Replacing (20), (22) and (23) into (27) and (28) and then substituting the resulting expressions into (30) and facilitating terms, yield
\[ \mathcal{T}(1) = \frac{2H(0)}{H(0) - H(-0^2)} \left\{ \int_{0}^{\infty} \left( \frac{a^2}{S^2} - y^{3-1} \right) v(f) \, dv \right\} - \frac{2H(-0^2)}{[H(0) - H(-0^2)]} \left[ \frac{a^2}{S^2} + h^3 \right] \frac{p}{[H(0) - H(-0^2)]}. \quad (31) \]

Substituting (20) and (31) into (16) and taking the inverse transform from the resulting expression, we have
\[ r_1^{1}(r, \theta) = \frac{1}{2\pi i} \int_{c_1}^{c_2} r^{5-1} \left[ \frac{H(0)}{H(0) - H(-0^2)} \frac{H(0)}{H(0) - H(-0^2)} \right] \left\{ \int_{0}^{\infty} \left( \frac{a^2}{S^2} - y^{3-1} \right) \, dv \right\} + \frac{2H(0)}{H(0) - H(-0^2)} \left[ \frac{a^2}{S^2} + h^3 \right] p \, ds. \quad (32) \]

Since Eq. (32) expresses a complex integral, the residual theorem is employed to compute the integral. So, a closed path is needed for integrating. The path of integration (line \( \text{Re} = C \)) should be completed by a semi-circular arc with infinite radius containing the negative part or positive part of the real axis. To find the poles, we may define
\[ R \cos \psi = \cos \theta - \frac{C_{45}}{C_{44}} \sin \theta, \quad \text{in which} \]
\[ \tan \psi = \frac{C_0 \sin \theta}{C_{44} \cos \theta - C_{45} \sin \theta}, \quad R^2 = \frac{C_{55} \sin^2 \theta + C_{44} \cos^2 \theta - C_{45} \sin 2\theta}{C_{44}}. \quad (33) \]

The new parameter \( \psi \) may be called the transformed apex angle of the anisotropic sector, which depends on the actual apex angle of the sector as well as material constants. Therefore, from (33) one can write
\[ R(\cos \psi + i \sin \psi) = R e^{i \psi} = \cos \theta + \left( -\frac{C_{45}}{C_{44}} + \frac{i C_0}{C_{44}} \right) \sin \theta. \quad (35) \]

which after using Eq. (12), it can be written as
\[ R e^{i \psi} = (\cos \theta + p \sin \theta). \quad (36) \]

According to Eqs. (36) and (14), we may write
\[ H(\theta) = (\cos \theta + p \sin \theta)^5 = R^5 e^{i5\psi}, \quad (37) \]
and also
\[ \overline{H}(\theta) = (\cos \theta + p \sin \theta)^5 = R^5 e^{-i5\psi}, \quad (38) \]
\[ H(-\theta) = (\cos \theta - p \sin \theta)^5 = R^5 e^{-i\psi}, \quad \overline{H}(-\theta) = (\cos \theta - p \sin \theta)^5 = R^5 e^{i\psi}, \quad (39) \]

where
\[ \tan \psi_1 = C_0 \sin \theta_1 \quad \frac{C_{44} \cos \theta_1 - C_{45} \sin \theta_1}{-C_0 \sin \theta_2} = \frac{C_{44} \cos \theta_2 + C_{45} \sin \theta_2}{C_{44}} = \frac{C_{55} \sin^2 \theta_1 + C_{44} \cos^2 \theta_1 - C_{45} \sin 2\theta_1}{C_{44}} = \frac{R_1^2}{C_{44}} \]
\[ \tan \psi_2 = \frac{C_{44} \cos \theta_2 + C_{45} \sin \theta_2}{C_{44} \cos \theta_2 - C_{45} \sin \theta_2} = \frac{C_{55} \sin^2 \theta_2 + C_{44} \cos^2 \theta_2 - C_{45} \sin 2\theta_2}{C_{44}} = \frac{R_2^2}{C_{44}} \quad (40) \]

Replacing following expressions into (32) and facilitating, yields
\[ r^{1,1}_{1,1}(r, \theta) = \frac{1}{2\pi i} \int_{c_1}^{c_2} \left\{ \int_{0}^{\infty} \frac{a^2}{S^2} - y^{3-1} \right\} v(f) \, dv \right\} \frac{H(0)}{H(0) - H(-0^2)} \left[ \frac{a^2}{S^2} + h^3 \right] p \, ds. \quad (41) \]
\[ \sin |\psi_1 - \psi_2| = 0 \quad \Rightarrow \quad S_n = \pm \frac{\pi}{\psi_1 - \psi_2} = \pm |\psi_1 - \psi_2|, \quad n = 0, 1, 2, 3 \]

To obtain the stress field, contour integration should be carried out. Both the integrands in (40) are meromorphic functions in \( S \) and four distinct regions of \( 0 \leq r \leq c_1, \quad c_1 \leq r \leq c_2, \quad c_2 \leq r \leq h \) and \( h \leq r \) should be considered. Since we need only the region \( c_1 \leq r \leq c_2 \) for applying the boundary condition (7), we may carry out the contour integration solely in this zone. For this reason, for the first integral in (40), after changing the order of integration we must break the limits of the integral in \([c_1, c_2] \) into regions \([c_1, r] \) and \([r, c_2] \). Then, for the integral in \([c_1, r] \) we break the expression in the parentheses to two terms and then complete the contour of integration for the first term by a semi-circular arc to include the negative part of the real axis, \( \text{Re}(S) < 0 \), and for the next term by a semi-circular arc to include the positive part of the real axis, \( \text{Re}(S) > 0 \). However for the integral in \([r, c_2] \) a semi-circular arc containing the negative part of the real axis, \( \text{Re}(S) < 0 \), must be consid-
er. For the second integral in (40), we complete the contour of integration by a semi-circular arc to include the negative part of the real axis, \( \text{Re}(S) < 0 \). Since the integrands in (40) vanish as \( |S| \to \infty \), by utilizing the residue theorem and applying the boundary condition (7), we obtain

\[
\sum_n \left( \frac{1}{(\psi_1 - \psi_2)} \right)^n \cdot R_1^{n \omega \sin(n \omega \psi_2)} - R_2^{-n \omega \sin(n \omega \psi_2)} \left( \frac{g^{2n \omega \sin(n \omega \psi_2)}}{h^{2n \omega \sin(n \omega \psi_2)}} + h^{-n \omega \sin(n \omega \psi_2)} \right)^m P
\]

\[- \sum_n \left( \frac{1}{(\psi_1 - \psi_2)} \right)^n \cdot \sin(n \omega \psi_2) \cdot \sum_n \left( \frac{1}{(\psi_1 - \psi_2)} \right)^n \cdot \sin(n \omega \psi_2)\]

\[
\times \int_{\psi_1}^{\psi_2} C_0 \left( \frac{V}{\alpha} \right)^m \cdot \left( \frac{V}{\alpha} \right)^n \cdot f(v) dv - \sum_n \left( \frac{1}{(\psi_1 - \psi_2)} \right)^n \cdot \sin(n \omega \psi_2) \cdot \sum_n \left( \frac{1}{(\psi_1 - \psi_2)} \right)^n \cdot \sin(n \omega \psi_2)\]

\[
\times \int_{\psi_1}^{\psi_2} C_0 \left( \frac{V}{\alpha} \right)^m \cdot \left( \frac{V}{\alpha} \right)^n \cdot f(v) dv = 0, \ c_1 \leq r \leq c_2.
\]

This is the basic relation for the derivation of the singular integral equation. Eq. (42) may be rewritten as follows

\[
\frac{P}{(\psi_1 - \psi_2)} \cdot \int \left( \frac{(-1)^n \rho_1 e^{2n \omega \psi_2} + \sum_{n=0}^{\infty} (-1)^n \rho_2 e^{2n \omega \psi_2}}{(-1)^n \rho_2 e^{2n \omega \psi_2} + \sum_{n=0}^{\infty} (-1)^n \rho_2 e^{2n \omega \psi_2}} \right)
\]

\[
= \frac{C_0}{\tau (\psi_1 - \psi_2)} \cdot \left( \int \sum_{n=0}^{\infty} (-1)^n \rho_1 e^{2n \omega \psi_2} f(v) dv - \sum_{n=0}^{\infty} (-1)^n \rho_1 e^{2n \omega \psi_2} f(v) dv \right)
\]

\[
+ \frac{C_0}{\tau (\psi_1 - \psi_2)} \cdot \left( \int \sum_{n=0}^{\infty} (-1)^n \rho_2 e^{2n \omega \psi_2} f(v) dv - \sum_{n=0}^{\infty} (-1)^n \rho_2 e^{2n \omega \psi_2} f(v) dv \right)
\]

\[
= \int_{\psi_1}^{\psi_2} C_0 \left( \frac{V}{\alpha} \right)^m \cdot \left( \frac{V}{\alpha} \right)^n \cdot f(v) dv = 0, \ c_1 \leq r \leq c_2.
\]

where

\[
\rho_1 = R_1^{n \omega \sin(n \omega \psi_2)} \left( \frac{1}{\beta} \right)^n \cdot \left( \frac{V}{\alpha} \right)^n; \ \rho_2 = R_1^{n \omega \sin(n \omega \psi_2)} \left( \frac{1}{\beta} \right)^n \cdot \left( \frac{V}{\alpha} \right)^n
\]

\[
= R_2^{-n \omega \sin(n \omega \psi_2)} \left( \frac{1}{\beta} \right)^n \cdot \left( \frac{V}{\alpha} \right)^n; \ \rho_4 = R_2^{-n \omega \sin(n \omega \psi_2)} \left( \frac{1}{\beta} \right)^n \cdot \left( \frac{V}{\alpha} \right)^n
\]

\[
= \left( \frac{V}{\alpha} \right)^m \cdot \left( \frac{V}{\alpha} \right)^n \cdot \rho_6 = \left( \frac{V}{\alpha} \right)^m \cdot \left( \frac{V}{\alpha} \right)^n; \ \rho_7 = \left( \frac{V}{\alpha} \right)^m \cdot \left( \frac{V}{\alpha} \right)^n
\]

\[
= \left( \frac{V}{\alpha} \right)^m \cdot \left( \frac{V}{\alpha} \right)^n.
\]

Considering the following series expansion formula:

\[
\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1 + x}; \ |x| < 1.
\]

Eq. (43) can be written in closed form as:

\[
\text{Eq. (43) becomes}
\]

\[
\int_{-\delta}^{\delta} \frac{x \phi(t)}{e^{-\frac{x}{\delta} - x}} dt = \frac{2\pi \phi(t)}{e^{-\frac{x}{\delta} - x}} + \int_{e^{\frac{x}{\delta} - x}}^{e^{\frac{x}{\delta} + x}} \frac{1}{e^{\frac{x}{\delta} - x} + 2e^{\frac{x}{\delta} + x}} d\phi(t)
\]

\[
= \int_{-\delta}^{\delta} \left( \frac{2 \phi(t)}{e^{-\frac{x}{\delta} - x}} + \int_{e^{\frac{x}{\delta} - x}}^{e^{\frac{x}{\delta} + x}} \frac{1}{e^{\frac{x}{\delta} - x} + 2e^{\frac{x}{\delta} + x}} d\phi(t)\right) dt
\]

\[
= \int_{e^{\frac{x}{\delta} - x}}^{e^{\frac{x}{\delta} + x}} \frac{1}{e^{\frac{x}{\delta} - x} + 2e^{\frac{x}{\delta} + x}} d\phi(t)
\]

\[
\text{It is seen that Eq. (52) has the form of a standard singular integral equation (Muskhelebshvili, 1953) as}
\]

\[
\text{Eq. (42) becomes}
\]

\[
\text{Eqs. (49) becomes}
\]

\[
\int_{-\delta}^{\delta} \frac{x \phi(t)}{e^{-\frac{x}{\delta} - x}} dt = \frac{2\pi \phi(t)}{e^{-\frac{x}{\delta} - x}} + \int_{e^{\frac{x}{\delta} - x}}^{e^{\frac{x}{\delta} + x}} \frac{1}{e^{\frac{x}{\delta} - x} + 2e^{\frac{x}{\delta} + x}} d\phi(t)
\]

\[
= \int_{e^{\frac{x}{\delta} - x}}^{e^{\frac{x}{\delta} + x}} \frac{1}{e^{\frac{x}{\delta} - x} + 2e^{\frac{x}{\delta} + x}} d\phi(t)
\]

\[
\text{Eqs. (52) becomes}
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\[
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\[
\text{Eqs. (52) becomes}
\]

\[
\text{Eqs. (52) becomes}
\]
\[ k(t,x) = -\frac{c^2}{\alpha(x^2 - t^2) + \left(c^4 + c^2k_\alpha x + t^2 + 2ck_\alpha x + t\right)} \]
\[ f(x) = 2\pi e \int \left( c^2R_{i\alpha}^\alpha d \cdot x \sin(\alpha\psi_1) + c^2R_{i\alpha}^\beta d \cdot x \cos(\alpha\psi_1) \right) \]
\[ - \left( c^2 + c^2k_\alpha x^2 + 2ck_\alpha x + t \right) \]
\[ \gamma \left( 1, 2, \ldots, n - 1 \right) \]
\[ \gamma \left( 1, 2, \ldots, n - 1 \right) \]

5. Determination of the stress intensity factors

After determining \( F(\eta) \), the stress intensity factors may be obtained by the following relations (Erdogan and Gupta, 1972):
\[ K(c_1) = \lim_{r \to c_1} \left( 2 - c_1 \right) \gamma_{i\alpha}(r, 0) \]
\[ K(c_2) = \lim_{r \to c_2} \left( 2 - c_2 \right) \gamma_{i\alpha}(r, 0) \]

In above relation, the method may be determined by means of Eq. (40). In order to take the location and the length of the crack into account, we may write the distance of the center of the crack with respect to the apex and the crack length in terms of the crack tips coordinates
\[ a_0 = \frac{c_2 - c_1}{2} \quad c_0 = \frac{c_2 + c_1}{2} \]
infinite large at the first crack tip, when \( \frac{a_0}{L} \) tends to unity. Nevertheless, Shahani (2001) showed that although \( K(c_1) \) has an increasing trend as \( \frac{a_0}{L} \) approaches the sector apex, it becomes zero when \( \frac{a_0}{L} = 1 \) (or \( c_1 = 0 \)). Another important point which can be observed from Fig. 3 is that by increasing the \( \frac{\psi_1 - \psi_2}{\pi} \) value, the value of the stress intensity factor increases. This is because of the interaction of the enhanced stress singularity at the sector apex, which is resulted from the higher value of \( \frac{\psi_1 - \psi_2}{\pi} \), and the crack tip stress singularity.

In Fig. 4, within all cases \( \theta_1 = \frac{\pi}{3}; \theta_2 = \frac{\pi}{2}; \psi_1 = \psi_2 = 2.00; \psi_1 = \frac{\pi}{3}; \psi_2 = \frac{\pi}{2}; \psi_1 = \frac{\pi}{2}; \psi_2 = \frac{\pi}{3}; (\psi_1 - \psi_2) = 2.77 \) and \( (\psi_1 - \psi_2) = 2.83 \) the \( \psi_1 \) value is less than \( \pi \). Thus, the singularity of the sector tip disappears. It is found that by decreasing \( \psi_1 \) value, the value of \( K(c_1) \) becomes greater than \( K(c_2) \). It is seen that the range of variations of the stress intensity factors become more limited compared with the case when \( \psi_1 > \psi_2 \). Moreover, \( K(c_1) \) does not essentially decrease when the first tip of the crack recedes the sector apex.

Fig. 5 shows the variation of the stress intensity factors as functions of relative crack distance, in the case of \( a_0 = 5 \) and \( a_0 = 4 \), for an apex angle of \( \theta_1 = \frac{\pi}{2} \) in different composite sectors, i.e., different material properties. Three pairs of elastic constants have been chosen as shown in the figure. It is observed that in all of the cases \( K(c_1) \), \( K(c_2) \) as \( \frac{a_0}{L} \) increases and also \( K(c_1) \) as \( \frac{a_0}{L} \rightarrow 1 \) (or \( c_1 = 0 \)), whereas \( K(c_2) \) is finite for \( \frac{a_0}{L} = 1 \) (or \( c_1 = 0 \)).

Figs. 6 and 7 show the variation of the stress intensity factors \( K(c_1) \) and \( K(c_2) \), respectively as functions of sector angle \( \theta_1 \) for
\[ h_1 + h_2 = \pi, \] in the case of \( a_0 = 5, b_0 = 2 \) and \( a_0 = 4, \)

for different composite sectors. The stress intensity factors have been obtained for similar configuration of \( h_1 + h_2 = \pi \), but for different material combinations.

Noting the above figures it can be concluded that:

1. The distribution of the stress intensity factor is symmetric in an isotropic sector \( C_{44}/C_{55} = 1, C_{45}/C_{55} = 0 \).

2. The distribution of the stress intensity factor is symmetric when \( C_{45}/C_{55} = 0 \). Three issues will be generated in terms of the value of \( C_{44}/C_{55} \) as follows:
   
   2.1 If \( C_{44}/C_{55} = 1 \), the maximum of the stress intensity factor occurs in \( \theta_1 = \theta_2 = \pi/2 \).
   
   2.2 If \( C_{44}/C_{55} = 0.5 \), the curve of the stress intensity factor becomes uniform in a wide domain, the value of which corresponds to the maximum value of the stress intensity factor.
   
   2.3 If \( C_{44}/C_{55} < 0.5 \), the curve of stress intensity factor gets two peaks.

3. When \( C_{45} \neq 0 \), the distribution of the stress intensity factor is not symmetric anymore and has just one peak. For a \( C_{45} \) with the same absolute value but with opposite sign, the related curve of the stress intensity factor is observed to be the mirror image with respect to the line \( \theta_1 = \pi/2 \).

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Fig. 4. Variations of the stress intensity factors as functions of relative crack distance for different sector angles.

Fig. 5. Variations of the stress intensity factors as functions of relative crack distance for a sector with different materials.
It should be emphasized that as the radial crack approaches the traction-free boundaries (that is, $h_1 \to (0, \pi)$), all of the stress intensity factors become zero as expected.

References


